Probability Exercise: Groups of Vaccinated Couples and Immunity

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Problem statement

Couples get together for a social gathering after all individuals are vaccinated for a disease. The vaccinations do not give 100% immunity. Couples with one or two individuals symptomatic for the disease do not go to the gathering. What is the chance that a member of some couple becomes infected by a member of another couple who is infected but non-symptomatic for the disease?

What is the distribution of the number of infections of this sort?

How many cross-couple infections are expected?

Assumptions must be stated to solve this problem

Number of couples (also size of "couples" – could be roommates)

Probability of immunity

Independence and homogeneity of individuals and couples

Probability of being infected and asymptomatic if not immune

Probability of infection if not immune and exposed

Pedagogical value

- Real-world problem – my parents asked me the question!
- Requires logic to reason through scenarios
- Highlights need to consider assumptions – one could contrast results under different assumptions
- Can start with two couples, then three, then four …
- Many smaller problems – ask questions sequentially
- Combinatorics, Binomial probability calculation, probability of an event not happening, multiplication rule, conditional probability, empty set, logic
- Opportunity for simulation game (use moderate probabilities, 0.3-0.7)
- Could program a simulation (more advanced – final project?)
- Significant digits

Example of assumptions

Number of couples: 4

Probability of immunity: 80%

- Assume this means no new infection.

Independence and homogeneity of individuals and couples

- Individuals, couples are similar – at least in terms of issues in this problem

Probability of being infected and asymptomatic if not immune: 10%

- This might be high for one encounter, but it could express the chance over several meetings.

Probability of infection if not immune and exposed to an infected person from another couple: 20% without masks

- If a both in a couple are not immune and one is infected, assume that the risk of infection by a spouse/partner is high and do not consider infection from another couple as an outcome of interest.

Binomial (n=8, p=0.80) for number immune
Binomial (n=8, p=0.20) for number *not* immune

# not immune	Prob # not immune	Prob # not immune and in 0 couples	Prob # not immune and in 1 couple	Prob # not immune and in 2 couples	Prob # not immune and in 3 couples	Prob # not immune and in 4 couples
8	**<0.0001**	0.0000	0.0000	0.0000	0.0000	**<0.0001**
7	**0.0001**	0.0000	0.0000	0.0000	0.0000	**0.0001**
6	**0.0011**	0.0000	0.0000	0.0000	**0.0002**	**0.0010***
5	**0.0092**	0.0000	0.0000	0.0000	**0.0039**	**0.0052***
4	**0.0459**	0.0000	0.0000	**0.0039**	**0.0315**	**0.0105**
3	**0.1468**	0.0000	0.0000	**0.0629**	**0.0839**	0.0000
2	**0.2936**	0.0000	**0.0419**	**0.2517**	0.0000	0.0000
1	**0.3355**	0.0000	**0.3355**	0.0000	0.0000	0.0000
0	**0.1678**	**0.1678**	0.0000	0.0000	0.0000	0.0000

* Some rows do not add to second column due to rounding.

Calculations – like Binomial but modify combinatorics, sometimes use subtraction

- P(2 not immune from the sample couple) = (4 C 1)*0.8^6*0.2^2 = 0.0419
- P(2 not immune and from 2 couples) = [(8 C 6) – (4 C 1)]*0.8^6*0.2^2
  = 0.2936 - 0.0419 = 0.2517 = [(4 C 2)*(2 C 1)*(2 C 1)] *0.8^6*0.2^2
- P(2 not immune and from 3 or 4 couples) = 0

- P(3 not immune from the same couple) = 0
- P(3 not immune and from 2 couples) = [(4 C 1)*(3 C 1)*(2 C 1)]* 0.8^5*0.2^3 = 0.0629
- P(3 not immune and from 3 couples) = [(4 C 3)*$(2 C 1)^3$]* 0.8^5*0.2^3 = 0.0839
- P(3 not immune and from 4 couples) = 0

Cross-couple infection risk, one example

2 infections in 2 couples:
Y = immune,
N = not immune

Couple 1	YY
Couple 2	YY
Couple 3	YN
Couple 4	YN

Probability 0.2516852

Overall probability:
0.2516852*0.036 = 0.009059697
≈ 0.009, or about 9 in 1000

Given this arrangement, are the two not immune infected?
I = infected, C = clear

NN	CC	
NN	C I	← cross-couple infect possible
NN	I C	← cross-couple infect possible
NN	I I	

P(CI or IC) = 2*0.1*0.9 = 0.18

Given this arrangement, probability of cross couple infection?

P(cross-couple infection) = 0.18*0.2 = 0.036

My simulation result

1 million replications

P(at least one cross infection) = 0.0312, about 3%

Expected number of cross infections = 0.034.

Number of individuals cross infected	0	1	2	3	4
Probability	0.968766	0.028220	0.002852	0.000158	0.000004
	0.969	0.028	0.003	<0.001	<0.001

Summary

- A suggestion for a probability exercise – plenty of room for refinement
- Real-world (mostly, personal risk)
- Complex but simpler sub-problems
- Highlights the role of logic and assumptions

Thanks.

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More calculations…

- P(4 not immune and from 2 couples) = [(4 C 2) * $0.8^4 * 0.2^4$ = 0.0039
- P(4 not immune and from 4 couples) = [(2 C 1 $)^4$* $0.8^4 * 0.2^4$ = 0.0105
- P(4 not immune and from 3 couples) = (8 C 4)* $0.8^4 * 0.2^4$ – 0.0039 – 0.0105 = 0.0315
- P(5 not immune and from 4 couples) = [(4 C 1)*(2 C 1$)^3$]* $0.8^3 * 0.2^5$ = 0.0052
- P(5 not immune and from 3 couples) = [(4 C 1)*(2 C 1)*(3 C 2)]* $0.8^3 * 0.2^5$ = 0.0039
- P(6 not immune and from 3 couples) = (4 C 3)* $0.8^2 * 0.2^6$ = 0.00016384 ≈ 0.0002
- P(6 not immune and from 4 couples) = [(4 C 2)*(2 C 1$)^2$]* $0.8^2 * 0.2^6$ = 0.00098304 ≈ 0.0010