

Beyond normal: Understanding power through R Shiny

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Motivation

Suppose that $X_i \stackrel{iid}{\sim} \text{Exp}(\theta)$, where θ represents the mean, and we take a random sample of size n from this population. Further suppose that we wish to test the following hypotheses:

$$\begin{aligned}H_0: \theta &\leq \theta_0 \\H_a: \theta &> \theta_0\end{aligned}$$

Consider the following test function:

$$\phi(\mathbf{X}) = \begin{cases} 1 & \sum_{i=1}^n X_i > k \\ 0 & \text{else} \end{cases},$$

where k is chosen such that $Pr(\phi(\mathbf{X}) = 1 | \theta_0) = \alpha$. It can be shown that if $X_i \stackrel{iid}{\sim} \text{Exp}(\theta)$, then $\sum_{i=1}^n X_i \sim \text{Gamma}(n, \theta)$. Let $T = \sum_{i=1}^n X_i \sim \text{Gamma}(n, \theta)$. To define our test, we seek the value k such that

$$Pr(\phi(\mathbf{X}) = 1 | \theta_0) = Pr(T > k | \theta_0) = 1 - Pr(T \leq k | \theta_0) = \alpha$$

The value of k that satisfies this equation is the $(1 - \alpha)^{th}$ quantile of the $\text{Gamma}(n, \theta_0)$ distribution. Letting $\Gamma_{n, \theta_0, 1-\alpha}$ denote this value, our test function becomes

$$\phi(X) = \begin{cases} 1 & \sum_{i=1}^n X_i > \Gamma_{n, \theta_0, 1-\alpha} \\ 0 & \text{else} \end{cases}$$

Therefore, our power function is

$$\text{Power}(\theta) = Pr(\mathbf{X} \in RR) = Pr\left(\sum_{i=1}^n X_i > \Gamma_{n, \theta_0, 1-\alpha} | \theta\right) = 1 - Pr\left(\sum_{i=1}^n X_i \leq \Gamma_{n, \theta_0, 1-\alpha} | \theta\right)$$

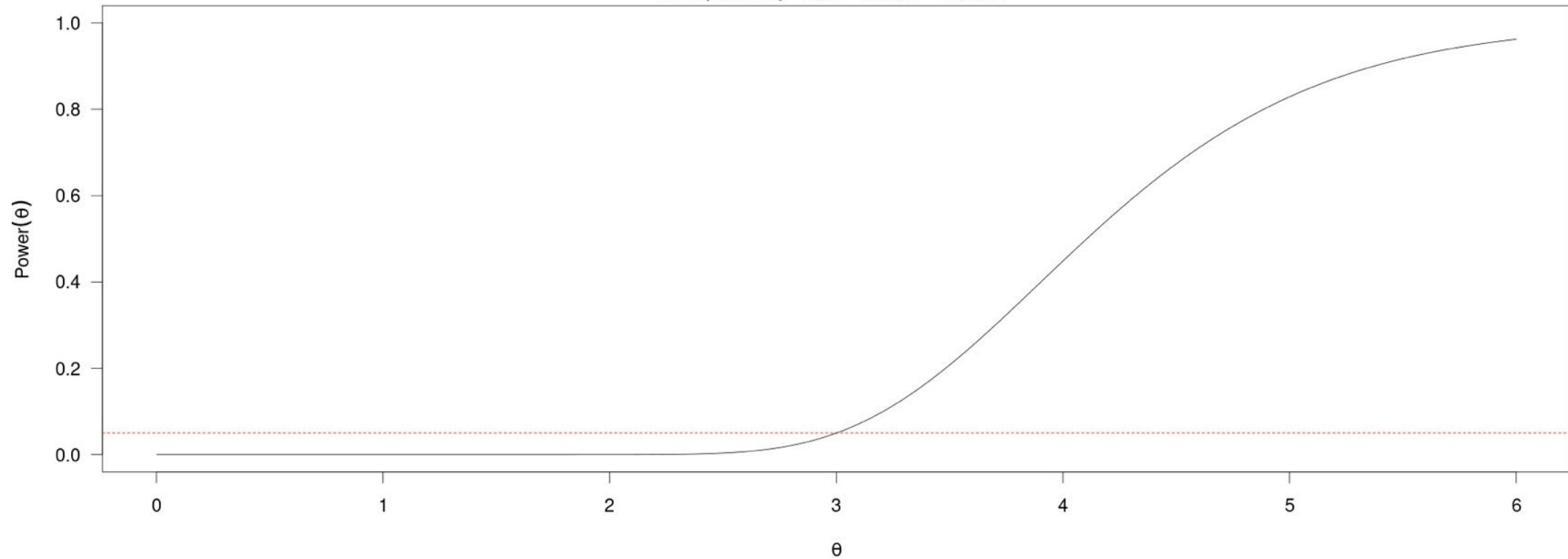
□

Motivation

$$\text{Power}(\theta) = \Pr(\mathbf{X} \in RR) = \Pr\left(\sum_{i=1}^n X_i > \Gamma_{n,\theta_0,1-\alpha} \mid \theta\right) = 1 - \Pr\left(\sum_{i=1}^n X_i \leq \Gamma_{n,\theta_0,1-\alpha} \mid \theta\right)$$

Power Function for $T(\mathbf{X}) = \sum(X_i)$ for samples from the Exponential distribution with mean θ

Plot options: $\theta_0 = 3$, $\alpha = 0.05$, $n = 25$, alt: >



Motivation

- But what does power really mean? How does it connect to other concepts such as sampling distributions?
- Shiny applets can be used to build intuitive relationships with power functions without dedicating class time to multiple derivations
 - Can be accompanied by an activity to focus the exploration

Application goal #1

Understanding how the sample size, null value, alternative hypothesis, significance level, test statistic, and population distribution each affect the power function

The Power of Sampling Distributions

Population Distribution

Exponential

Test Statistic (T(X))

Sum of the X's

Null Value (θ_0)

3

Alternative Hypothesis

Greater than

Alpha Level (α)

0.05

Sample Size (n)

25

Theta (θ)

Plots

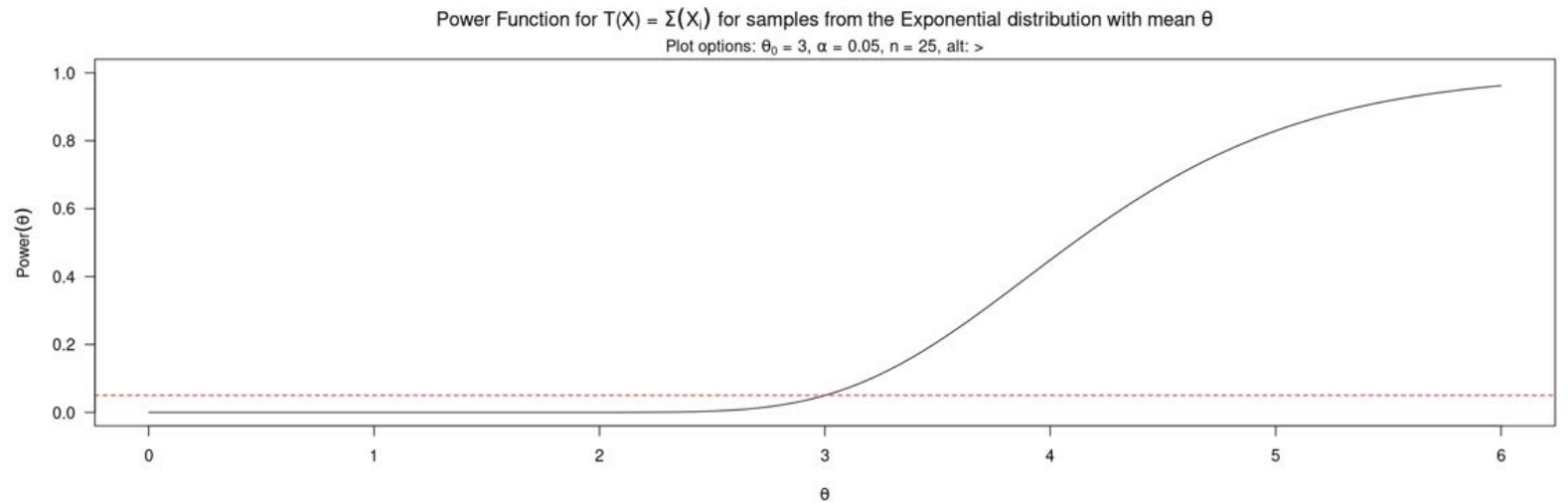
Derivations

Introduction

Hello! This application is meant to help visualize power curves, sampling distributions, and the relationship between the two. The panel to the left will allow you to explore the power curve for each of three different population distributions, alternative hypotheses, and test statistics. Furthermore, you may investigate how those curves depend on the significance level, sample size and null value.

This application will also allow you to explore how power is related to the sampling distribution of a test statistic under the null hypothesis and true value of θ . To visualize these distributions for a specific value of θ , simply click the power curve at the desired value of θ , or type the value in the panel. Have fun!

Power Curve



Download this plot

Sampling Distribution

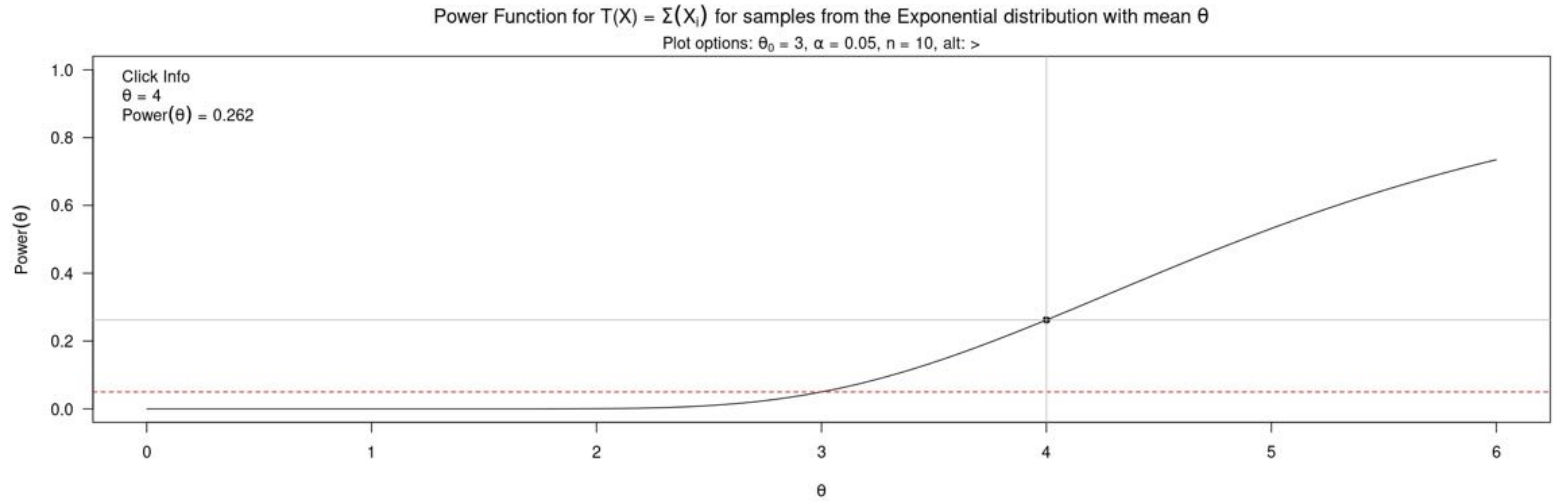
Below are the sampling distributions for the chosen statistic under both the null hypothesis and the true value of θ . The red area corresponds to the significance level and the gray area corresponds to the power; note the relationship between the sampling distribution under θ_0 , sampling distribution under θ , significance level and power!

Click a point on the power plot above to visualize the sampling distribution!

Application goal #2

Understanding the relationship between the power function and the sampling distribution of the test statistic under both the null hypothesis and the true value of θ

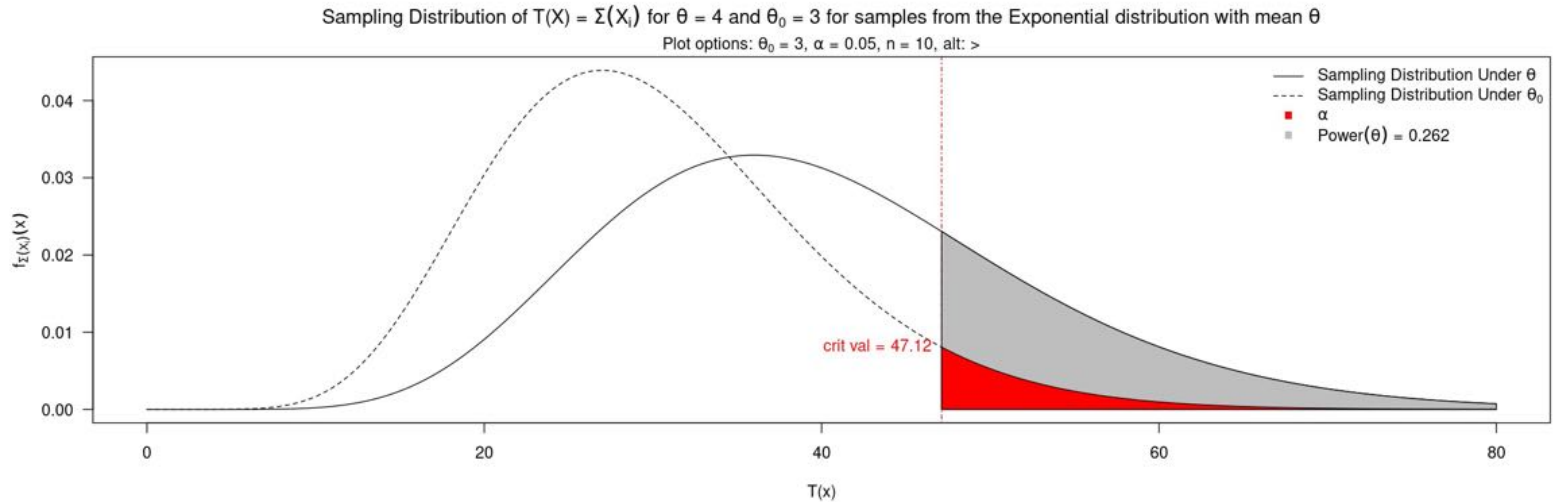
Power Curve



Download this plot

Sampling Distribution

Below are the sampling distributions for the chosen statistic under both the null hypothesis and the true value of θ . The red area corresponds to the significance level and the gray area corresponds to the power; note the relationship between the sampling distribution under θ_0 , sampling distribution under θ , significance level and power!



Population Distribution

Exponential

Test Statistic (T(X))

Sum of the X's

Null Value (θ_0)

3

Alternative Hypothesis

Greater than

Alpha Level (α)

0.05

Sample Size (n)

10

Theta (θ)

4

Application goal #3

Understanding how to calculate the power function of a hypothesis test using a variety of strategies, including simulation and normal approximation

Number of simulated samples

Sample Size (n)

Null Value (θ_0)

Alpha Level (α)

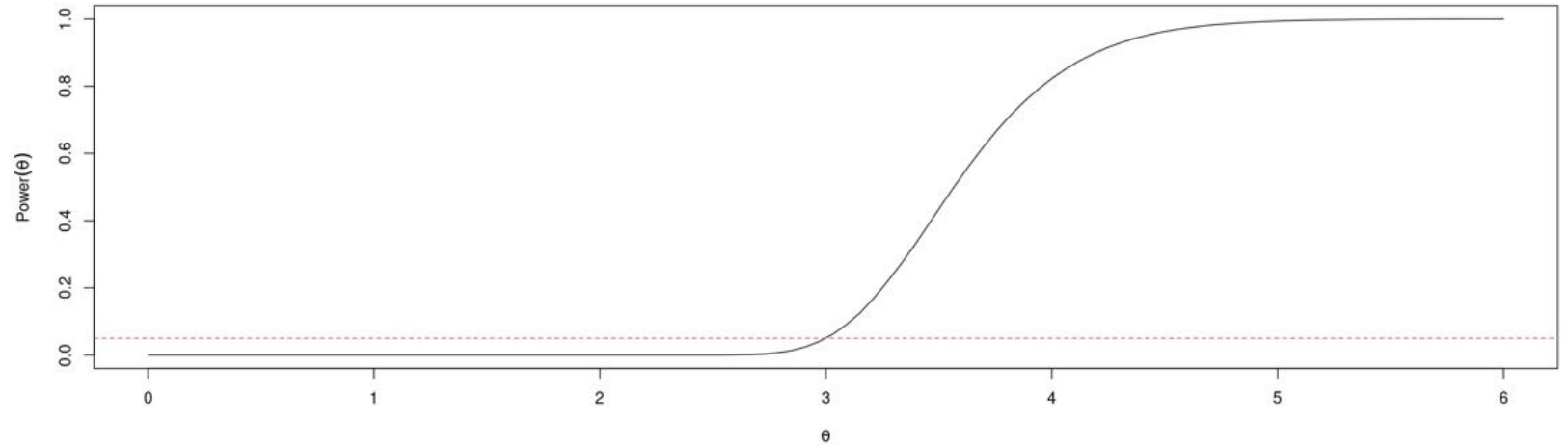
Alternative Hypothesis

Theta (θ)

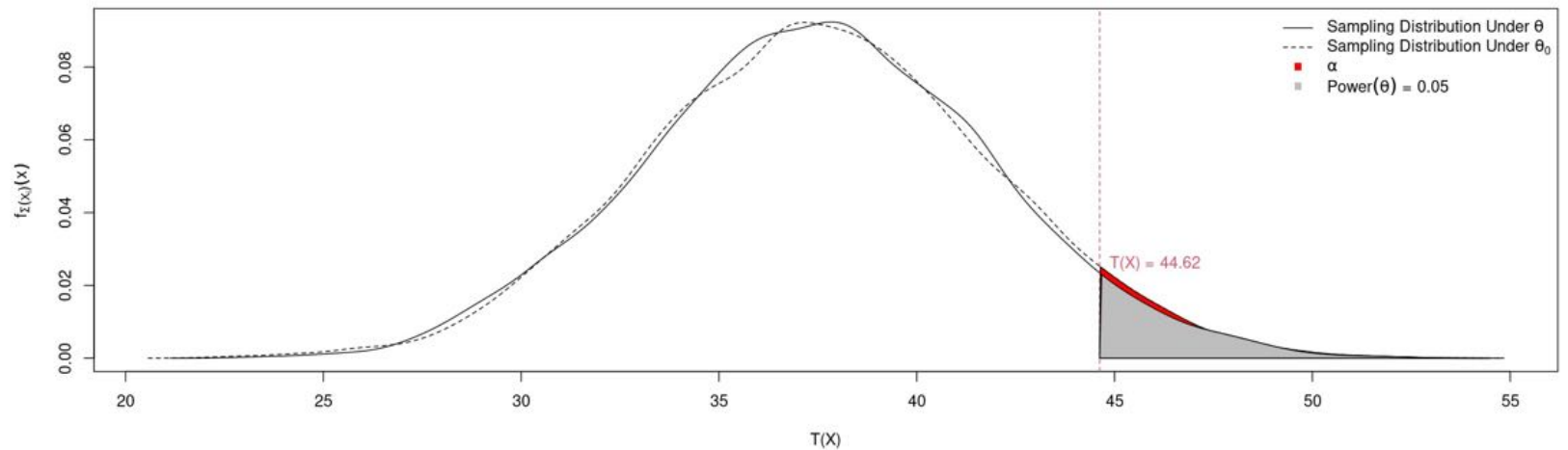
Plots Derivations Irwin-Hall Normal Approximation **Simulated Power**

The purpose of this tab is to calculate and visualize power for each alternative hypothesis for the sum of uniform random variables through simulation. To do so, choose the appropriate values from the side panel and click 'Simulate power!' Please note that for large numbers of simulated samples, it will take time to compute.

Simulated Power Function for $T(X) = \Sigma(X_i)$



Simulated Sampling Distribution for $T(X) = \Sigma(X_i)$ for $\theta = 3$ and $\theta_0 = 3$



Let's explore!

Using the applet, reflect on how students would engage with the applet and activity.

Relevant links:

- Web application link: <https://shiny.stt.msu.edu/jg/powerapp>
- Activity link: [Power exploration activity](#)

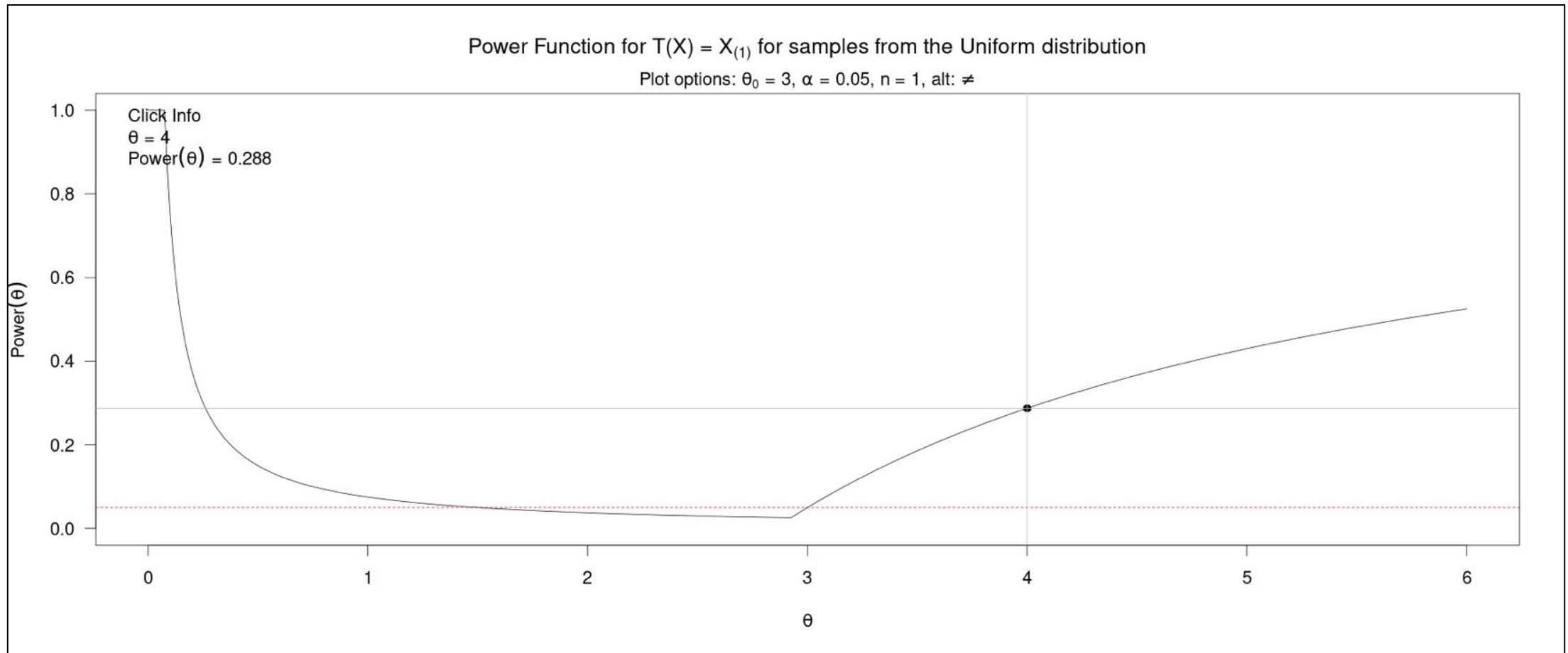
Discussion

- What did you discover and notice?
- How might you use this applet in your classroom?

Additional student insights

- Biased vs unbiased tests
 - uniform distribution, not equal to alternative hypothesis, sample size of one
- Significance level and p-values
- Sufficiency and power
- Numerical instability
 - uniform distribution, sum of random variables

Biased test



Numerical instability

Population Distribution
Uniform

Test Statistic (T(X))
Sum of the X's

Null Value (θ_0)
3

Alternative Hypothesis
Greater than

Alpha Level (α)
0.05

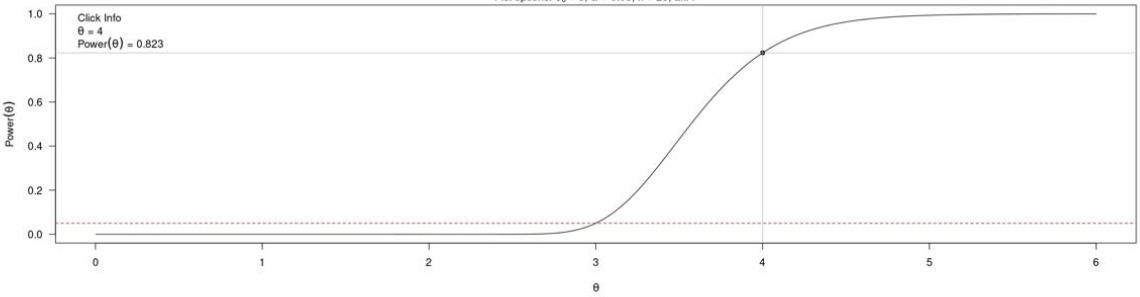
Sample Size (n)
25

Theta (θ)
4

Use normal approximation

Power Curve

Power Function for $T(X) = \Sigma(X)$ for samples from the Uniform distribution
Plot options: $\theta_0 = 3$, $\alpha = 0.05$, $n = 25$, alt: >

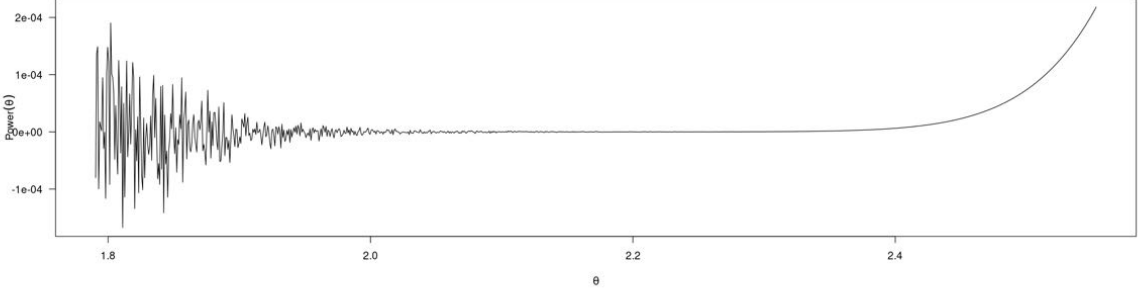


Download this plot

Click here to hide warning information.

- Numerical Instability:** This power function (and the resulting sampling distributions) exhibits strange oscillating behavior due to numerical instability in the Irwin-Hall distribution function. This problem is exacerbated and more noticeable as the sample size increases; see Alberto (2019) for greater detail. The plot below allows you to better visualize the numerical instability.

Power Function for $T(X) = \Sigma(X)$ for θ in (1.79, 2.55)



- Central Limit Theorem:** One option to remedy this numerical instability issue is to use the Central Limit Theorem to approximate the sampling distribution of the sum of uniform random variables. For sample sizes of even four or greater, the Central Limit Theorem provides a reasonably good approximation to the sampling distribution. Use the 'Irwin-Hall Normal Approximation' tab to further investigate this relationship. To use the CLT to approximate the power function and sampling distributions, check the 'Use normal approximation' box on the side panel.
- Simulation:** Another option often used in practice is to simulate the power. To do so, we generate draws from the sampling distribution of the sum of uniform random variables under both the null value and true value of theta. We then choose our critical value(s) such that we reject with probability alpha when the null hypothesis is true by pulling quantiles from the empirical CDF of the simulated sampling distribution under the null value. Finally, we calculate the proportion of simulated statistics under theta that were as or more extreme than these critical values, which becomes our estimate of the power. To see this in action, click the 'Simulated Power' tab.

Summary

- Shiny applets can be used to build intuitive relationships with power functions without dedicating class time to multiple derivations
- Use of this web application expands students' opportunities to actively explore and develop conceptual understanding of statistical concepts

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